Quick Review

A continuous RV X has a probability density function $f_X(x)$ that satisfies

$$\mathbb{P}[a < X < b] = \int_a^b f_X(x)\, dx$$

PDFs satisfy the same properties as PMFs:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\forall x, \quad f_X(x) \geq 0$$

and we can define $\mathbb{E}$ and Var:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$\mathrm{Var}(X) = \underbrace{\int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx}_{= \mathbb{E}[X^2]} - \underbrace{\left[\int_{-\infty}^{\infty} x f(x)\, dx \right]^2}_{= \mathbb{E}[X]^2}$$

An RV also has a cumulative distribution function

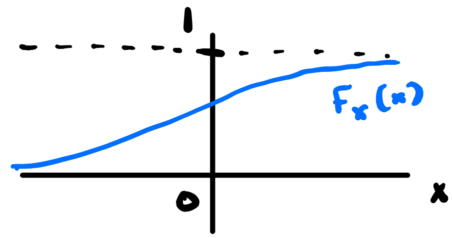
$$F_X(x) = \mathbb{P}[X \leq x]$$

that satisfies

(1) $\lim_{x \to -\infty} F_X(x) = 0$

(2) $\lim_{x \to \infty} F_X(x) = 1$

(3) $F_X(x)$ is strictly nondecreasing



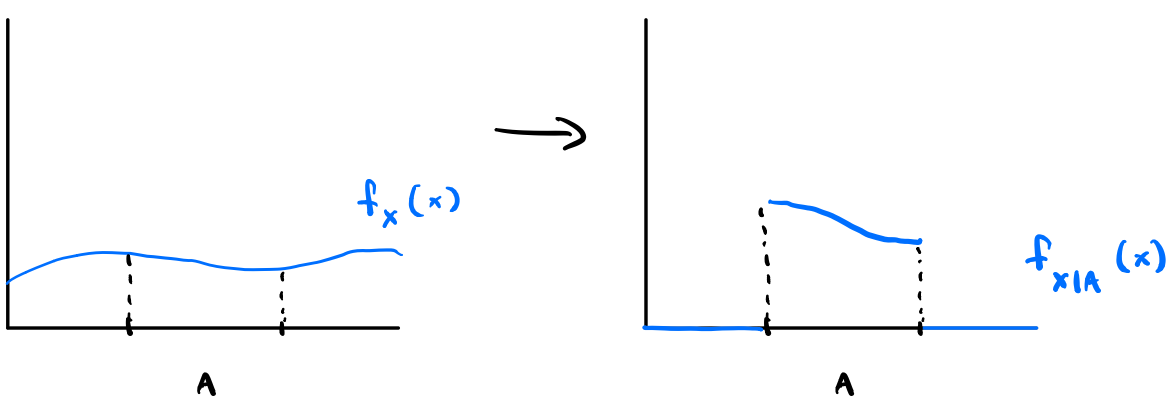
MOST IMPORTANT: If X has PDF f_X and CDF F_X, then

$$f_X(x) = \frac{d}{dx} F_X(x)$$

We can also do conditional probability:

Let A be an event. Then the conditional PDF of X given A, $f_{X|A}(x)$, is

$$f_{X|A}(x) = \begin{cases} f_X(x) / \mathbb{P}[A] & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$



1 Condition on an Event

The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2}, & \text{if } 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

2 Max of Uniforms

Let $X_1, ...X_n$ be independent $U[0,1]$ random variables, and let $X = \max(X_1, ...X_n)$. Compute each of the following in terms of n.

(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

3 Darts but with ML

Suppose Alice and Bob are playing darts on a circular board with radius 1. When Alice throws a dart, the distance of the dart from the center is uniform $[0,1]$. When Bob throws the dart, the location of the dart is uniform over the whole board. Let X the a random variable corresponding to the distance of the player's dart from the board.

(a) What is the pdf of X if Alice throws

(b) What is the pdf of X if Bob throws

(c) Suppose we let Alice throw the dart with probability p, and let Bob throw otherwise. What is the pdf of X (your answer should be in terms of p)?

(d) Using the same premise as in part c, suppose you observe a dart on the board but don't know who threw it. Let x be the dart's distance from the center. We would like to come up with a decision rule to determine whether Alice or Bob is more likely to have thrown the dart given your observation, x. Specifically, if we let A be the event that Alice threw the dart and B be the event that Bob threw, we want to guess A if $\mathbb{P}[A|X \in [x, x+dx]] > \mathbb{P}[B|X \in [x, x+dx]]$ (what do these two probabilities have to sum up to?). For what values of x would we guess A? (your answer should be in terms of p)