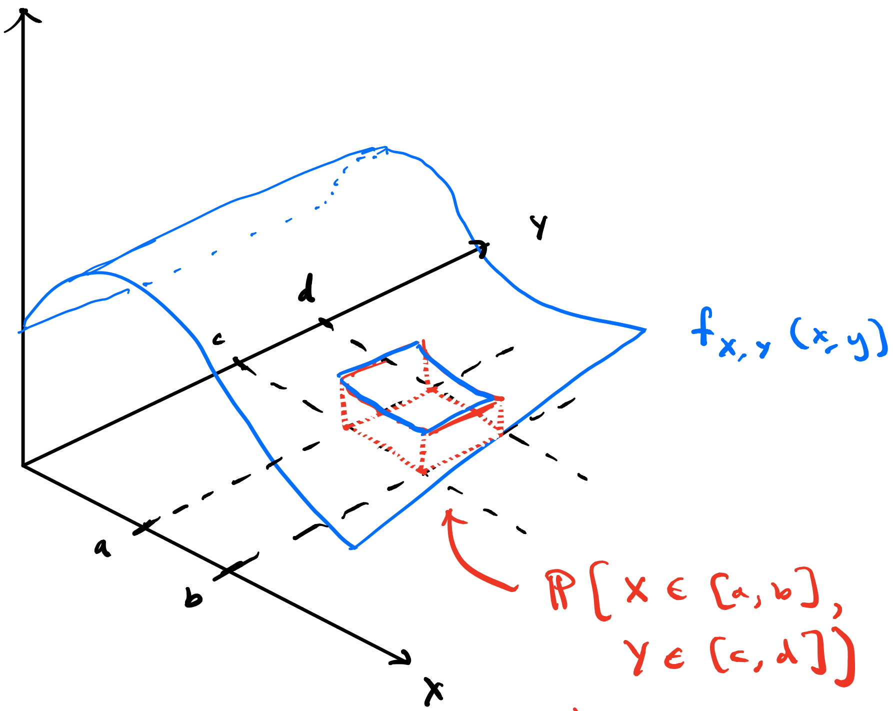
Quick Review

- Two continuous RVs X, Y have joint density $f_{X,Y}(x,y)$ that satisfies

$$\mathbb{P}[X \in [a,b], Y \in [c,d]] = \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

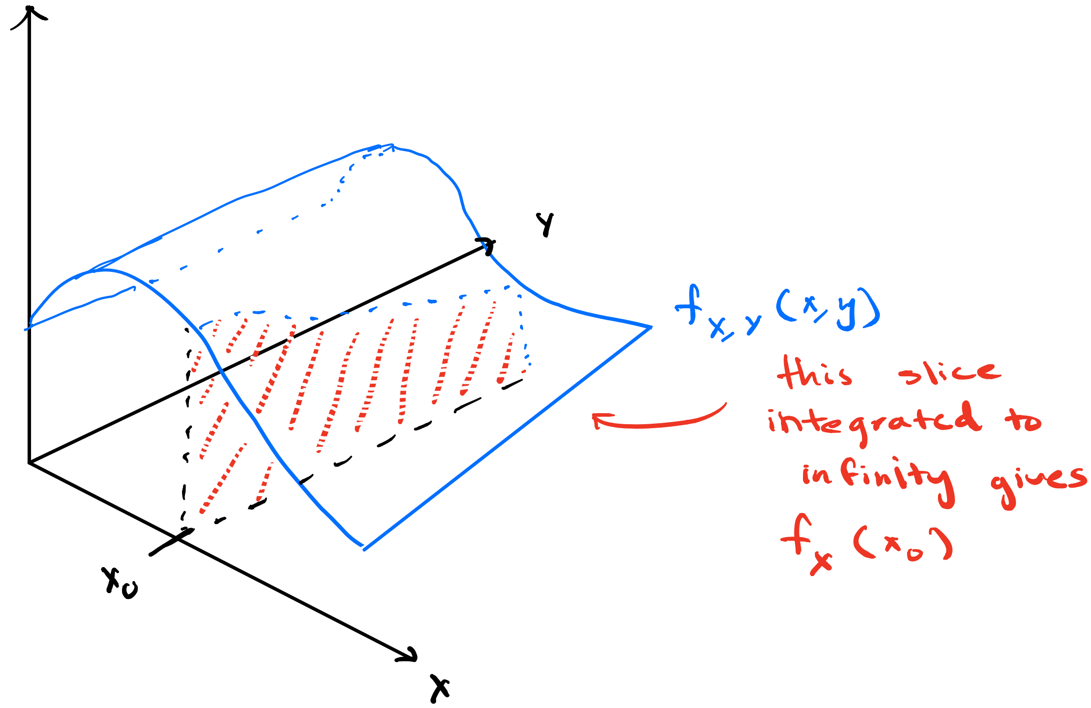


$\mathbb{P}[X \in [a,b], Y \in [c,d]]$ is this volume under $f_{X,Y}(x,y)$.

- We can isolate a marginal density f_X via integration:

$$f_X(x_0) = \int_{-\infty}^{\infty} f_{X,Y}(x_0, y)\, dy$$

same constant value

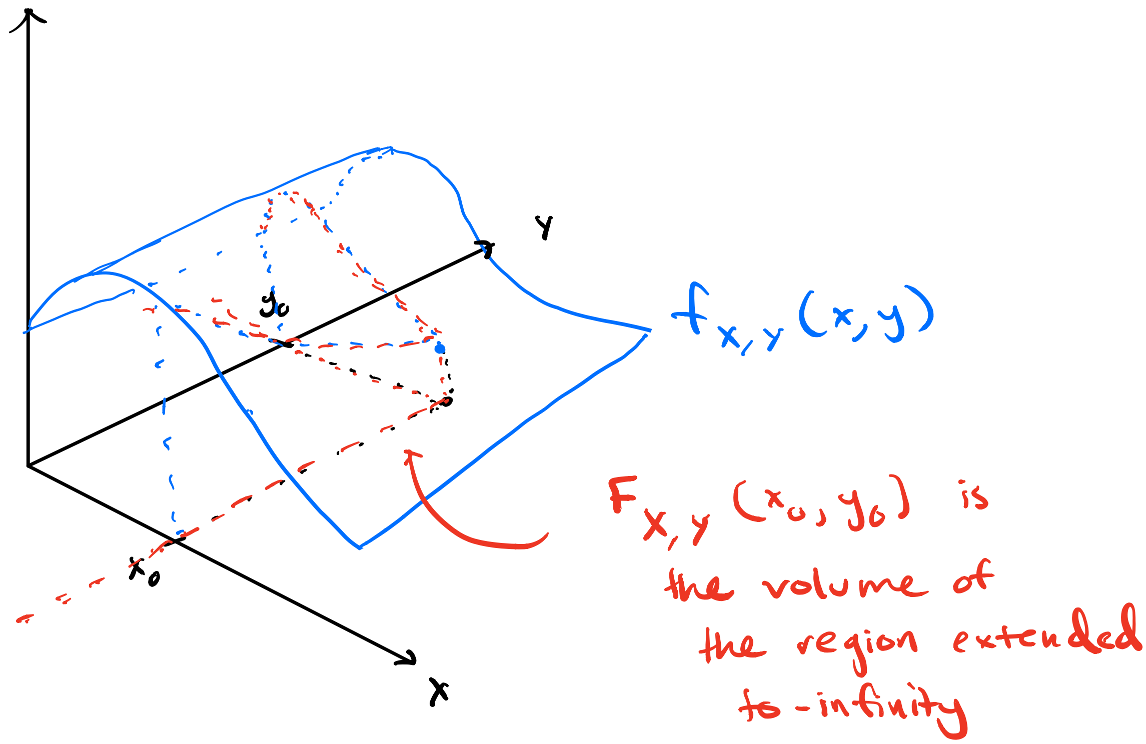


- We can also define the joint CDF of X, Y by

$$F_{x,y}(x,y) = \mathbb{P}[X < x, Y < y]$$

↳ $$F_{x,y}(x_0, y_0) = \int_{-\infty}^{x_0} \int_{-\infty}^{y_0} f_{x,y}(x,y) \, dy \, dx$$

↳ $$f_{x,y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{x,y}(x,y)$$



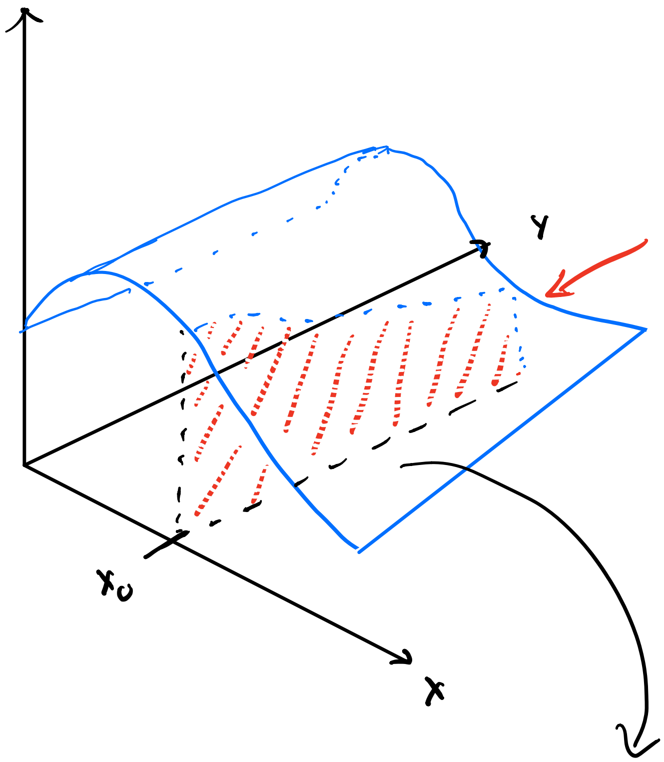
- We can also condition one continuous RV on another.

$$f_{Y|X}(y|x_0) = \frac{f_{x,y}(x_0, y)}{f_x(x_0)}$$

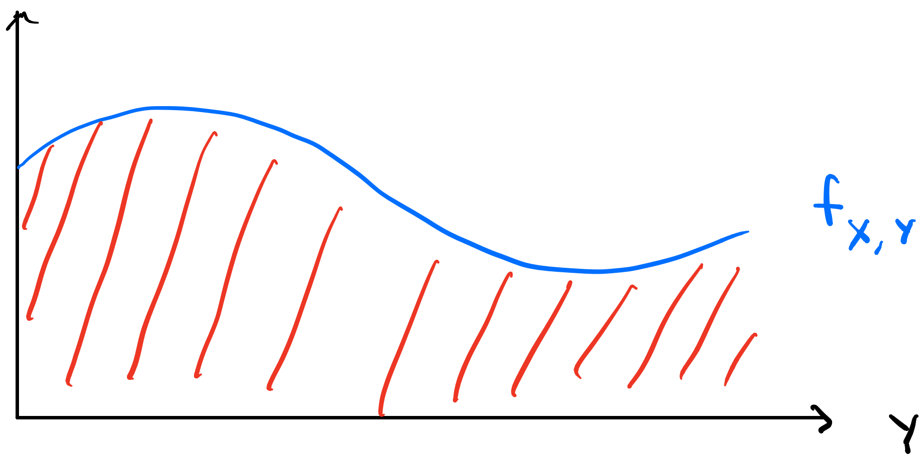
$$= \frac{f_{x,y}(x_0, y)}{\int_{-\infty}^{\infty} f_{x,y}(x_0, y)\, dy}$$

Area is $f_X(x_0)$

$f_{X,Y}(x,y)$



restrict to
$X = x_0$ (1)



Normalize by
dividing by (2)
$f_X(x_0)$

- X, Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

Reference:

Total Probability:

	Discrete	Continuous
D:	$\mathbb{P}[A] = \sum_i \mathbb{P}[A \mid B_i]\, \mathbb{P}[B_i]$	$\mathbb{P}[A] = \int_{-\infty}^{\infty} \mathbb{P}[A \mid X = x]\, f_X(x)\, dx$
C:	$f_X(x) = \sum_i f_{X \mid B_i}(x)\, \mathbb{P}[B_i]$	$f_X(x) = \int_{-\infty}^{\infty} f_Y(y)\, f_{X \mid Y}(x \mid y)\, dy$

Bayes' Theorem

	Discrete	Continuous
D:	$\mathbb{P}[A_i \mid B] = \dfrac{\mathbb{P}[A_i]\, \mathbb{P}[B \mid A_i]}{\sum_j \mathbb{P}[A_j]\, \mathbb{P}[B \mid A_j]}$	$\mathbb{P}[A_i \mid X = x] = \dfrac{\mathbb{P}[A_i]\, f_{X \mid A_i}(x)}{\sum_j \mathbb{P}[A_j]\, f_{X \mid A_j}(x)}$
C:	$f_{X \mid A}(x) = \dfrac{f_X(x)\, \mathbb{P}[A \mid X = x]}{\int_{-\infty}^{\infty} f_X(t)\, \mathbb{P}[A \mid X = t]\, dt}$	$f_{X \mid Y}(x \mid y) = \dfrac{f_X(x)\, f_{Y \mid X}(y \mid x)}{\int_{-\infty}^{\infty} f_X(t)\, f_{Y \mid X}(y \mid t)\, dt}$

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1 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \le x \le 1, 0 \le y \le 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal distribution of X.

(c) Find the conditional distribution of Y given $X = x$.

(d) Are X and Y independent?

2 Uniform Distribution

You have two spinning wheels, each having a circumference of 10 cm with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinning wheel's mark and Y be the position of the second spinning wheel's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

3 Exponential Practice

Let $X \sim \text{Exponential}(\lambda_X)$ and $Y \sim \text{Exponential}(\lambda_Y)$ be independent, where $\lambda_X, \lambda_Y > 0$. Let $U = \min\{X,Y\}$, $V = \max\{X,Y\}$, and $W = V - U$.

(a) Compute $\mathbb{P}(U > t, X \leq Y)$, for $t \geq 0$.

(b) Use the previous part to compute $\mathbb{P}(X \leq Y)$. Conclude that the events $\{U > t\}$ and $\{X \leq Y\}$ are independent.

(c) Compute $\mathbb{P}(W > t \mid X \leq Y)$.

(d) Use the previous part to compute $\mathbb{P}(W > t)$.

(e) Calculate $\mathbb{P}(U > u, W > w)$, for $w > u > 0$. Conclude that U and W are independent. [*Hint*: Think about the approach you used for the previous parts.]