Review:

- Probability Formalism
  - ↳ Goal: use <u>set theory</u> to rigorize our discussion of probability
  - ↳ Random Experiment: any scenario w/ uncertainty, e.g.
    - Coin toss
    - Sequence of dice rolls
    - Outcome of an election where every voter votes randomly.
  - ↳ Any RE can be defined with the following
    - A <u>sample space</u> Ω, which is a set of all possible outcomes.
    - A function $\mathbb{P} : \Omega \to \mathbb{R}$ satisfying
      - $\mathbb{P}[\omega] \in [0,1] \ \forall \omega \in \Omega$, and
      - $\sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1$.
  - ↳ An <u>event</u> in an RE is a subset $E \subseteq \Omega$
    - Can typically be thought of "all outcomes with a certain property".
    - e.g., if the RE is two coin flips, $E = \{(H,H), (T,T)\}$ is "the event that the two flips are the same".
    - Can define $\mathbb{P}[E] = \sum_{e \in E} \mathbb{P}[e]$.

- Principle of Inclusion - Exclusion (PIE)
  - ↳ Idea: If you have bunch of sets A_i that overlap with each other, how do you figure out how many elements total across all of them?
  - ↳ Solution: iteratively correct for over/under counting in the following way:

$$\left|\bigcup_i A_i\right| = \sum_i |A_i| - \sum_{i_1 \neq i_2} |A_{i_1} \cap A_{i_2}| + \sum_{i_1 \neq i_2 \neq i_3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}|$$

$$- \cdots + (-1)^{n-1} \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} |A_{i_1} \cap \cdots \cap A_{i_n}|$$

  - ↳ Each term $\sum_{i_1 \neq \cdots \neq i_k} |A_{i_1} \cap \cdots \cap A_{i_k}|$ corrects for under/overcounting.
    - If x is in k sets, it will be overcounted by the first sum, undercounted by the second, etc, until it hits the k^{th} term.

- Stars + Bars (Balls + Bins, Balls + Urns, ...)
  - ↳ Problem: How many ways can I split k indistinguishable objects among n people?
  - ↳ Solution: Consider the following sequence of "stars" and "bars", where we have k stars and $n-1$ bars:

$$* \; * \; | \; * \; * \; * \; | \; * \; \cdots \; * \; | \; * \; * \; *$$

↑ Person 1 gets 2 ↑ Person 2 gets 3 ↑ Person n gets 3

There are k stars, $n-1$ bars, so $n+k-1$ symbols total $\Rightarrow$ total # of sequences is $\binom{n+k-1}{n-1}$.

↳ Can be rephrased "k from n w/ replacement where order doesn't matter."

- Derangements:

↳ A <u>derangement</u> of length n is a permutation π of $[1, 2, \ldots, n]$ with <u>no fixed points</u>, e.g. ~~the~~ $\forall i,\ \pi(i) \neq i$.

↳ Can be counted using complement + PIE!

↳ Lets count all the permutations w/ <u>at least one fixed point</u>

- Let $A_i = \{\pi \mid \pi(i) = i\}$, i.e. the permutations that fix i.
- We want $\left|\bigcup_i A_i\right|$: all permutations with at least one fixed point. We can use PIE!

↳ $\left|\bigcup_i A_i\right| = \sum |A_i| - \sum |A_{i_1} \cap A_{i_2}| + \cdots$

↳ What does $|A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}|$ look like?

- The set of all permutations that fix $i_1, i_2, \ldots, i_k$ — so there are $(n-k)!$ of them!

↳ $\left|\bigcup_i A_i\right| = \sum_i n! - \sum_{i_1 \neq i_2} (n-1)! + \sum_{i_1 \neq i_2 \neq i_3} (n-2)! - \cdots$

$=$

$$\hookrightarrow \left| \bigcup_i A_i \right| = \sum_i (n-1)! - \sum_{i_1 \neq i_2} (n-2)! + \sum_{i_1 \neq i_2 \neq i_3} (n-3)! - \cdots$$

$$= n! - \frac{n!}{2!} + \frac{n!}{3!} - \frac{n!}{4!} + \cdots$$

$$= n! \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{1}{i!} = \overline{D}_n$$

Thus, the # of derangements is

$$D_n = n! - \overline{D}_n = n! \sum_{i=0}^{n} (-1)^i \cdot \frac{1}{i!} .$$