Discussion 1A Slides

CS 70 Summer 2020

June 23, 2020

Logistics

- This discussion is Monday - Thursday, 5-6pm Pacific
- If you are not signed up for this discussion, you are welcome to attend, but I will not record your attendance
- Feel free to email me (akamat@berkeley.edu)/post on Piazza if you have any questions/concerns

Quick Review

- Propositions are statements that are either true or false,
- Operators:
  - $\wedge$: "Conjunction" /"AND"
  - $\vee$: "Disjunction" /"OR"
  - $\neg$: "Negation" /"NOT"
  - $\implies$: "Implication" /"If, then"
  - $\iff$: "Biconditional" /"If, and only if"
- Quantifiers:
  - $\forall$: "Universal" /"For all"
    e.g. $\forall x \in \mathbb{R}\ x^2 > 0$.
  - $\exists$: "Existential" /"There exists"
    e.g. $\exists x \in \mathbb{N}\ x > 9000$.

Implication (Dis 1A Problem 1)

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x, y)$ that would make the implication false).

(a) $\forall x \, \forall y \; P(x, y) \implies \forall y \, \forall x \; P(x, y)$.
(b) $\exists x \, \exists y \; P(x, y) \implies \exists y \, \exists x \; P(x, y)$.
(c) $\forall x \, \exists y \; P(x, y) \implies \exists y \, \forall x \; P(x, y)$.
(d) $\exists x \, \forall y \; P(x, y) \implies \forall y \, \exists x \; P(x, y)$.

Implication (Dis 1A Problem 1)

(a) Is it true that $\forall x \, \forall y \; P(x, y) \implies \forall y \, \forall x \; P(x, y)$?

Answer: This implication is true.

Solution: These statements are actually the same thing - the first part in words is *for all* x *and for all* y, $P(x, y)$ *is true* and the second part in words is *for all* y *and for all* x, $P(x, y)$ *is true*.

Implication (Dis 1A Problem 1)

(b) Is it true that $\exists x \, \exists y \; P(x, y) \implies \exists y \, \exists x \; P(x, y)$?

Answer: This implication is true.

Solution: Again, these statements are actually the same thing - the first part in words is *there is an* x *and there is a* y *such that* $P(x, y)$ *is true* and the second in words is *there is a* y *and there is an* x *such that* $P(x, y)$ *is true*.

Implication (Dis 1A Problem 1)

(c) Is it true that $\forall x \, \exists y \; P(x, y) \implies \exists y \, \forall x \; P(x, y)$?

Answer: This implication is false.

Solution: In words, the first statement is *for all x, there is a y such that $P(x, y)$ is true*, while the second is *there exists a y such that for all x, $P(x, y)$ is true*. These are not the same thing!

The first statement is just that every x has a corresponding y that makes $P(x, y)$ true, while the second says that theres a 'special' y that makes $P(x, y)$ true no matter which x we pick.

Here's the counterexample - let x, y be real numbers and let $P(x, y) \equiv x < y$. Then it is true that for all real numbers x, we can find a real number y such that $x < y$, but we can't find a y that is larger than every possible real number x.

Implication (Dis 1A Problem 1)

(d) Is it true that $\exists x \, \forall y \; P(x,y) \implies \forall y \, \exists x \; P(x,y)$?

Answer: This implication is true.

Solution: In words, the first statement is *there exists an x such that for all y, $P(x,y)$ is true,* and the second statement is *for all y, there exists an x such that $P(x,y)$ is true.*

Again, these aren't the same thing, but in this case, the first implies the second. If the first statement is true, then theres a special x that makes $P(x,y)$ true for any choice of y, so for any y we pick, we know we can just pick the special x to make $P(x,y)$ true, hence the second statement is also true.

XOR (Dis 1A Problem 2)

The truth table of XOR (denoted by $\oplus$) is as follows.

A	B	A $\oplus$ B
F	F	F
F	T	T
T	F	T
T	T	F

(a) Express XOR using only $(\wedge, \vee, \neg)$ and parentheses.

(b) Does $(A \oplus B)$ imply $(A \vee B)$? Explain briefly.

(c) Does $(A \vee B)$ imply $(A \oplus B)$? Explain briefly.

XOR (Dis 1A Problem 2)

(a) Express XOR using only $(\wedge, \vee, \neg)$ and parentheses.

Solution: There are a lot of possible answers:

- $A \oplus B \equiv (A \wedge \neg B) \vee (\neg A \wedge B)$
- $A \oplus B \equiv (A \vee B) \wedge (\neg A \vee \neg B)$
- $A \oplus B \equiv (A \vee B) \wedge \neg(A \wedge B)$

XOR (Dis 1A Problem 2)

(b) Does $A \oplus B$ imply $A \vee B$?

Solution: Yes - if $A \oplus B$ is true, then exactly one of A, B are true, so $A \vee B$ is also true.

XOR (Dis 1A Problem 2)

(c) Does $A \vee B$ imply $A \oplus B$?

Solution: No - if both A and B are true, then $A \vee B$ is true but $A \oplus B$ is not.

Converse and Contrapositive (Dis 1A Problem 4)

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The converse of $P \implies Q$ is $Q \implies P$)

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$)

Converse and Contrapositive (Dis 1A Problem 4)

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

Solution: The statement is

$$\forall n \in \mathbb{N}\,(4 \mid n \implies 2 \mid n).$$

This is true, as if $4 \mid n$, then $n = 4k = 2(2k)$ for some integer k, hence n is divisible by 2.

Converse and Contrapositive (Dis 1A Problem 4)

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$)

Solution: The inverse is

$$\forall n \in \mathbb{N} \, (\neg(4 \mid n) \implies \neg(2 \mid n)),$$

which, in words, is the statement *if a natural number is not divisible by 4, then it is not divisible by 2.* This is false, as $n = 2$ is a counterexample.

Converse and Contrapositive (Dis 1A Problem 4)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The converse of $P \implies Q$ is $Q \implies P$)

Solution: The converse is

$$\forall n \in \mathbb{N} \, (2 \mid n \implies 4 \mid n),$$

which, in words, is the statement *if a natural number is divisible by* 2*, then it is divisible by* 4*.* Again this is false with $n = 2$ being a counterexample.

Converse and Contrapositive (Dis 1A Problem 4)

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$)

Solution: The contrapositive is

$$\forall n \in \mathbb{N}\ (\neg(2 \mid n) \implies \neg(4 \mid n)),$$

which, in words, is the statement *if a natural number is not divisible by* 2*, then it is not divisible by* 4. This is true, as there is no way for $n/4$ to be an integer if $n/2$ is not an integer.