Discussion 1B Slides

CS 70 Summer 2020

June 26, 2020

Warm-Up

If P and Q are both propositions, and we know that $P \implies Q$ and $\neg P \implies \neg Q$, does it follow that $P \iff Q$?

Solution: Yes! Since $P \implies Q$, we know that Q is true when P is true. Furthermore, since $\neg P \implies \neg Q$, we know that Q is false when P is false. Hence, they always have the same value, so $P \iff Q$.

Quick Review

Proof Techniques to show that $P \implies Q$:

- Direct Proof - Assume P holds, and then deduce that Q is also true.
- Contrapositive - Assume Q doesn't hold, and then deduce that P doesn't hold.
- Contradiction - Assume P holds and Q doesn't, then show that something breaks because of it.

Miscellany/General Problem Solving:

- You can show that something exists by constructing an example.
- You can exploit symmetry (if it exists) in your problem to reduce the amount of work you have to do.
- Half of the battle is asking the right questions!

Prove or Disprove (Dis 1B Problem 1)

Prove or disprove the following statements:

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$, then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Here $\mathbb{Z}^+$ denotes the set of positive inte)

Prove or Disprove (Dis 1B Problem 1)

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

Solution: This statement is true. The idea here is to pursue a direct proof. We want to show that n being odd implies that $n^2 + 4n$ is also odd. Factoring, we see that $n^2 + 4n = n(n + 4)$. Now, since n is odd, $n + 4$ is also odd, so $n(n + 4)$ is the product of two odd numbers, and thus also odd.

Prove or Disprove (Dis 1B Problem 1)

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$, then $a \leq 11$ or $b \leq 4$.

Solution: This statement is true. The idea here is to prove the contrapositive. Assume that $a > 11$ and $b > 4$ (this is the negation of "$a \leq 11$ or $b \leq 4$"); then we can add these two inequalities together to see that $a + b > 15$. This is exactly what we wanted to prove, as it is the negation of $a + b \leq 11$.

Prove or Disprove (Dis 1B Problem 1)

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

Solution: This statement is also true. The idea is again to use contraposition. Suppose r is rational. Then $r = \frac{a}{b}$ for some integers a, b (with $b \neq 0$). Squaring both sides, we can see that $r^2 = \frac{a^2}{b^2}$, which is the ratio of two integers. Hence, r^2 is also rational.

Prove or Disprove (Dis 1B Problem 1)

(d) $(\forall n \in \mathbb{Z}^+)\ 5n^3 > n!$. (Here $\mathbb{Z}^+$ denotes the set of positive inte)

Solution: This statement is false. To prove this, we provide a counterexample - consider $n = 10$. Then $5n^3 = 5000$ but $n! = 3628800$, which is significantly larger.

Pigeonhole Principle (Dis 1B Problem 2)

Prove the following statement: If you put $n+1$ balls into n bins (however you want, as long as every ball is in a bin), then at least one bin must contain at least two balls.

Solution: We'll go for a contradiction here. Assume that we have $n+1$ balls and have thrown them into the bins such that each bin has at most 1 ball. Then, since there are n bins, this means that there are at most n balls in total, which is a contradiction because we know that we have $n+1$ balls.

Numbers of Friends (Dis 1B Problem 3)

Prove that if there $n \geq 2$ people at a party, then two of them will always have the same number of friends at the party (assume friendships are reciprocated, i.e. that if Alice is friends with Bob, then Bob is friends with Alice and vice versa). Hint: you can use the result we proved in the previous problem here!

Numbers of Friends (Dis 1B Problem 3)

Solution: This is a pretty tricky problem, and the idea here is that we want to use the Pigeonhole Principle to show that there must be two people with the same number of friends.

In order to apply it, we need to identify our balls and our bins first. In this case, the people in the party are the balls (because we want to show that two people have the same number of friends) and the bins are the numbers 0 to $n-1$. So, a person is put into bin i if they have i friends. We still aren't done however; we have n balls and n bins - we need to find a way to reduce the number of bins.

The final tricky part of this proof is the following observation: if a person at the party has $n-1$ friends, then its impossible for anyone at the party to have 0 friends (why?). Because of this, at any point, at most $n-1$ bins are filled (since bin 0 and bin $n-1$ can't be filled simultaneously), so we can use the PHP to see that there must be 2 people in the same bin, i.e. two people with the same number of friends at the party.