Problem 1 (Spring 2019 Final 8.2)**:** Consider continuous random variables X, Y with joint density function $f_{X,Y}(x, y) = cxy$ for all $x, y \in [0, 1]$ and 0 everywhere else.

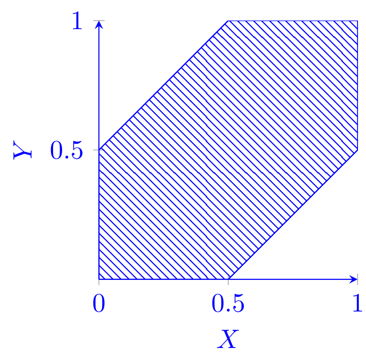
(a) What is the value of c?

(b) What is $\mathbb{P}\left[|X - Y| \leq \frac{1}{2}\right]$?

Solution 1: We know that

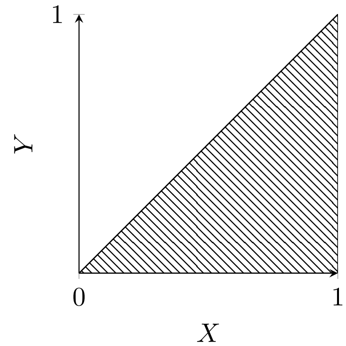
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\ dx\ dy = \int_0^1\int_0^1 cxy\ dx\ dy = 1,$$

so $c = 4$. For part (b), we just need to identify the region over which $|X - Y| \leq \frac{1}{2}$ and integrate $f_{X,Y}$ over that region. This region is shown below. We can then compute



$$\mathbb{P}\left[|X - Y| \leq \frac{1}{2}\right] = \int_0^{\frac{1}{2}}\int_0^{x+\frac{1}{2}} 4xy\ dy\ dx + \int_{\frac{1}{2}}^1\int_{x-\frac{1}{2}}^1 4xy\ dy\ dx = \boxed{\frac{41}{48}}$$

Problem 2 (Spring 2018 Final 5.14)**:** Consider continuous random variables X, Y with uniform joint density over the region $\{(x, y) \mid 0 \leq y < x \leq 1\}$ (shown below).



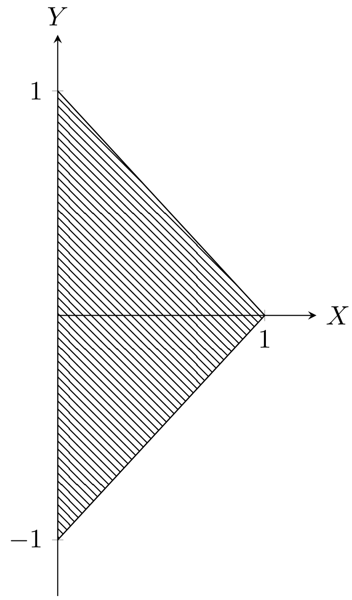
Suppose someone takes a sample of either X or Y with equal probability, then announces the value is $\frac{2}{3}$. What is the probability that the sample is from X?

Solution 2: See here.

Problem 3 (Summer 2019 Final 2w)**:** Let X, Y be independent uniform random variables over the interval $[0, 1]$. What is the CDF of $|X - Y|$?

Solution 3: See here.

Problem 4 (Summer 2019 Final 4 (Edited))**:** Suppose we have a triangle ABC in the plane with coordinates $A = (0.1), B = (1, 0)$, and $C = (0, -1)$, (see the figure below) and we choose a point uniformly at random from this triangle.



Let X be the random variable corresponding to the x coordinate of the point chosen, and let Y be the random variable corresponding to its y coordinate.

(a) Find the joint density $f_{X,Y}(x, y)$.

(b) Find $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(c) Let $Z = |X| + |Y|$. Find the PDF of Z.

Solution 4: See here.

Problem 5 (Spring 2017 Final 7.2)**:** You pick a real number from the range $[0, 1]$ using the uniform distribution. Then Alvin independently picks a real number uniformly at random from the range $[0, 2]$.

(a) What is the probability that your numbers differ by no more than 1?

(b) Suppose you pick your number from the same range except now with PDF $f_X(x) = 2x$, with Alvin still picking uniformly from $[0, 2]$. What is the probability that your numbers differ by no more than 1?

Solution 5*:* See Here.

Problem 6 (Spring 2016 Final 3.5)**:** You play a game of darts with a friend. You are better than they are and the distances of your shots from the center of the dartboard are i.i.d. $U[0,1]$ while theirs are $U[0,2]$. To make the game fair, you agree to throw one dart, while your friend throws two. The dart closest to the center wins the game. What is the probability that you win the game?

Solution 6: Let X denote the distance of your throw from the center (so $X \sim U[0,1]$) and let Y be the minimum of the two dart distances that your friend threw. We want to find $\mathbb{P}[X < Y]$. We have that $f_X(x) = 1$ for $x \in [0,1]$ and since Y is the minimum of two i.i.d. uniform RVs, we have $f_Y(y) = 1 - \frac{y}{2}$ for $y \in [0,2]$. Since X and Y are independent, their joint distribution is then

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = 1 - \frac{y}{2},$$

so our desired probability is just

$$\mathbb{P}[X < Y] = \int_0^1 \int_x^2 1 - \frac{y}{2}\, dy\, dx = \boxed{\frac{7}{12}}$$

Problem 7 (Fall 2018 Final 8)**:** For $n \geq 2$, let $X_1, \dots, X_n$ be independent $U[0,1]$ random variables and for $i \in \{1, \dots, n\}$, let Y_i denote the ith smallest value of $\{X_1, \dots, X_n\}$. For example, $Y_1 = \min\{X_1, \dots, X_n\}$ and $Y_n = \max\{X_1, \dots, X_n\}$.

(a) Find the PDF of Y_2.

(b) If $n = 2$, find the joint PDF of Y_1 and Y_2.

(c) Assume again that $n = 2$ and let $G = Y_2 - Y_1$ be the gap size between Y_1 and Y_2. Find the PDF of G.

(d) What is $\mathbb{P}\left[G > \frac{1}{2}\right]$?

Solution 7: In order to find the PDF of Y_2, we first find the CDF. The probability that $Y_2 > y$ for some $y \in [0,1]$ is the probability that all of $X_i > y$ plus the probability that exactly one $X_i < y$. Thus, we have that

$$\mathbb{P}[Y_2 > y] = \mathbb{P}[X_i > y \forall i] + \mathbb{P}[\text{Exactly } 1 \; X_i < y] = (1-y)^n + ny(1-y)^{n-1},$$

hence

$$F_{Y_2}(y) = 1 - \mathbb{P}[Y_2 > y] = \begin{cases} 0 & \text{if } y < 0, \\ 1 - (1-y)^n - ny(1-y)^{n-1} & \text{if } 0 \leq y < 1, \\ 1 & \text{if } y \geq 1. \end{cases}$$

Now, we can find the PDF using differentiation:

$$f_{Y_2}(y) = \frac{d}{dy} F_{Y_2}(y) = \begin{cases} n(n-1)y(1-y)^{n-2} & \text{if } 0 \leq y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For part (b), we first need to consider the support of the joint PDF. Since by definition, $0 \leq Y_1 \leq Y_2 \leq 1$, our PDF must be nonzero only on pairs (y_1, y_2) where $0 \leq y_1 \leq y_2 \leq 1$. With this understood, we now find the joint CDF first. Some casework shows that we have

$$\mathbb{P}[Y_1 < y_1, Y_2 < y_2] = \mathbb{P}[X_1 < y_1, X_2 < y_1] + \mathbb{P}[X_1 < y_1, y_1 < X_2 < y_2] + \mathbb{P}[y_1 < X_1 < y_2, X_2 < y_1],$$

so

$$F_{Y_1,Y_2}(y_1, y_2) = \begin{cases} y_1^2 + 2y_1(y_2 - y_1) & \text{if } 0 < y_1 < y_2 < 1, \\ 0 & \text{otherwise,} \end{cases}$$

thus taking the iterated partial derivative gives

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 2 & \text{if } 0 < y_1 < y_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For part (c), observe that $G = Y_2 - Y_1 = |X_2 - X_1|$. Now, we can find the CDF of G just by integrating the region $|X_2 - X_1| < g$ for $g \in [0,1]$. This gives us a CDF of

$$F_G(g) = \begin{cases} 1 - (1-g)^2 & \text{if } 0 \leq g < 1, \\ 0 & \text{if } g < 0 \\ 1 & \text{otherwise,} \end{cases}$$

so

$$f_G(g) = \begin{cases} 2 - 2g & \text{if } 0 \leq g \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For part (d), we can simply integrate the PDF of G from $\frac{1}{2}$ to 1 to get $\mathbb{P}\left[G > \frac{1}{2}\right] = \frac{1}{4}$.