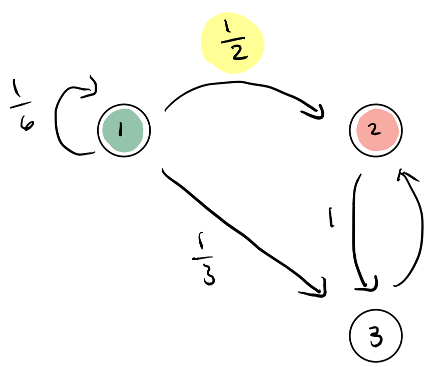
We can model discrete time systems where the future depends only on the present using Markov Chains.

- A Markov Chain is a sequence of random variables $X_0, X_1, \ldots$ over a common state space $\mathcal{X}$ satisfying the Markov Property:

$$\mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n, \underbrace{X_{n-1} = x_{n-1}, \ldots}] = \mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n]$$

Path to get to the present
is irrelevant for the future.

- X_i represents the "state" of the system at time i.
- We denote $P(i,j) = \mathbb{P}[X_{n+1} = j \mid X_n = i]$ to be the transition probability from state i to state j.
- We can succinctly describe a MC using a state space diagram and a transition matrix:



$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

- In order to fully specify a MC, we also need to provide an initial distribution:

$$\pi_0 = [\pi_0(1) \quad \pi_0(2) \quad \cdots]$$

$$= \mathbb{P}[X_0 = 2]$$

- Given the initial distribution, we can find the distribution of X_n via matrix multiplication:

$$\pi_n = [\pi_n(1) \cdots] = \pi_0 P^n$$

- A distribution π is stationary if it satisfies

$$\pi = \pi P, \quad \pi \mathbb{1} = 1$$

- A MC is irreducible if for any two states i, j, one can reach j from i.
  - ↳ If a MC is irreducible, then it has a unique stationary distribution.
- The period of state i in an irreducible Markov Chain is defined as

$$d(i) = \gcd \{ n > 0 \mid P^n(i,i) > 0 \}$$

  - ↳ $\forall i, j \in \mathcal{X}$, $d(i) = d(j)$.
  - ↳ If $d(i) = 1$, then we say the chain is aperiodic.
  - ↳ If a chain is aperiodic, it reaches its stationary distribution in the limit, no matter what initial distribution we start with.
- Fix some state $j \in \mathcal{X}$. We let $\beta(i)$ denote the hitting time of state j from state i. These satisfy

$$\beta(j) = 0,$$

$$\beta(i) = 1 + \sum_{k \in \mathcal{X}} P(i,k)\, \beta(k) \quad \forall i \neq j.$$

- Let $A, B \subset \mathcal{X}$ be disjoint. Let $\alpha(i)$ be the probability that we reach A before B, starting at i. These satisfy

$$\alpha(i) = 0 \quad \text{if } i \in B$$
$$\alpha(i) = 1 \quad \text{if } i \in A$$
$$\alpha(i) = \sum_{k \in \mathcal{X}} P(i,k)\,\alpha(k) \quad \text{o.w.}$$

Reference:

Balance Eqs: (stationary dist.)

$$\pi = \pi P, \quad \pi \mathbb{1} = 1$$

First Step Eqs: (hitting times)

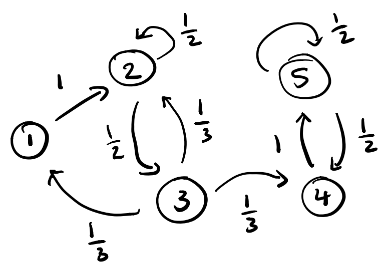
$$\beta(j) = 0$$
$$\beta(i) = 1 + \sum_{k \in \mathcal{X}} P(i,k)\,\beta(k) \quad \forall i \neq j$$

A before B:

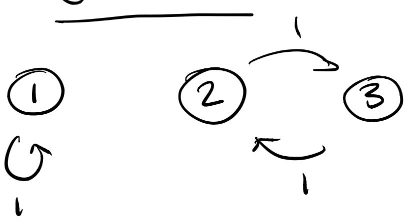
$$\alpha(i) = 0 \quad \forall i \in B$$
$$\alpha(i) = 1 \quad \forall i \in A$$
$$\alpha(i) = \sum_{k \in \mathcal{X}} P(i,k)\,\alpha(k) \quad \text{o.w.}$$

1. For each of the following chains, determine
(i) reducibility / irreducibility
(ii) periodicity / aperiodicity if it applies
(iii) stationary distribution if it is unique

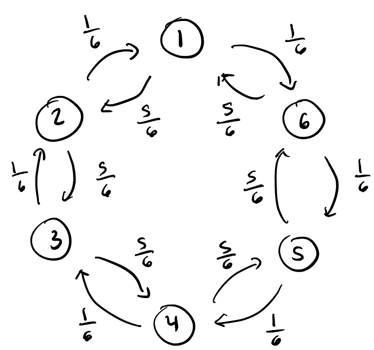
Chain 1



Chain 2



Chain 3



2. You flip a fair coin. What is the average number of tails until you get {tails, heads, heads}? (Spring 2016 Final 3.3)

3. Customers are in line at a store. At any given time step, the customer at the front of the line leaves with probability p, and, independent of that, a new customer joins the line with probability q. The line can only handle 4 people at most, so no new customer will join if there are already 4 customers present in the line. If there are 2 customers currently in line, what is the probability the line is empty before the line contains 4 people?