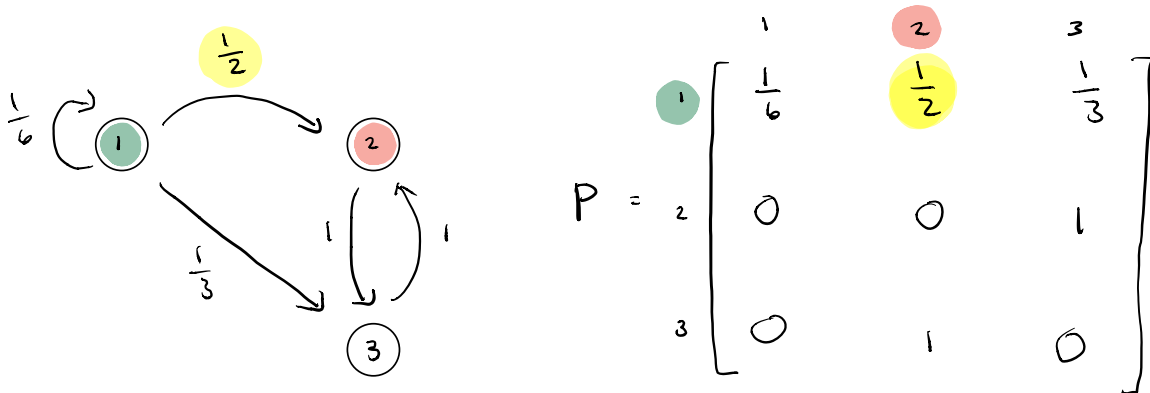


We can model discrete time systems where the future depends only on the present using **Markov Chains**.

- A **Markov Chain** is a sequence of random variables $X_0, X_1, \dots$ over a common **state space** $\mathcal{X}$ satisfying the **Markov Property**:

$$P[X_{n+1}=x_{n+1} \mid X_n=x_n, \underbrace{X_{n-1}=x_{n-1}, \dots}_{\text{Path to get to the present is irrelevant for the future.}}] = P[X_{n+1}=x_{n+1} \mid X_n=x_n]$$

- X_i represents the "state" of the system at time i .
- We denote $P(i, j) = P[X_{n+1}=j \mid X_n=i]$ to be the **transition probability** from state i to state j .
- We can succinctly describe a MC using a **state space diagram** and a **transition matrix**:



- In order to fully specify a MC, we also need to provide an **initial distribution**:

$$\pi_0 = [\pi_0(1) \quad \pi_0(2) \quad \dots]$$

$\uparrow$
 $= P[X_0 = 2]$

- Given the initial distribution, we can find the distribution of X_n via matrix multiplication:

$$\pi_n = [\pi_n(1) \cdots] = \pi_0 P^n$$

- A distribution π is **stationary** if it satisfies

$$\pi = \pi P, \pi \mathbf{1} = 1$$

- A MC is **irreducible** if for any two states i, j , one can reach j from i .

↳ If a MC is irreducible, then it has a unique stationary distribution.

- The **period** of state i in an irreducible Markov chain is defined as

$$d(i) = \gcd \{ n > 0 \mid P^n(i, i) > 0 \}$$

↳ $\forall i, j \in X, d(i) = d(j)$.

↳ If $d(i) = 1$, then we say the chain is **aperiodic**.

↳ If a chain is aperiodic, it reaches its stationary distribution in the limit, no matter what initial distribution we start with.

- Fix some state $j \in X$. We let $\beta(i)$ denote the **hitting time** of state j from state i . These satisfy

$$\beta(j) = 0,$$

$$\beta(i) = 1 + \sum_{k \in X} P(i, k) \beta(k) \quad \forall i \neq j.$$

- Let $A, B \subset X$ be disjoint. Let $\alpha(i)$ be the probability that we reach A before B , starting at i . These satisfy

$$\alpha(i) = 0 \quad \text{if } i \in B$$

$$\alpha(i) = 1 \quad \text{if } i \in A$$

$$\alpha(i) = \sum_{k \in X} P(i, k) \alpha(k) \quad \text{o.w.}$$

Reference:

Balance Eqs: $\pi = \pi P$, $\pi \mathbf{1} = 1$
(stationary dist.)

First Step Eqs:
(hitting times)

$$\beta(j) = 0$$

$$\begin{aligned} \beta(i) &= 1 + \sum_{k \in X} P(i, k) \beta(k) \\ &= \sum_{k \in X} P(i, k) (\beta(k) + 1) \end{aligned}$$

A before B:

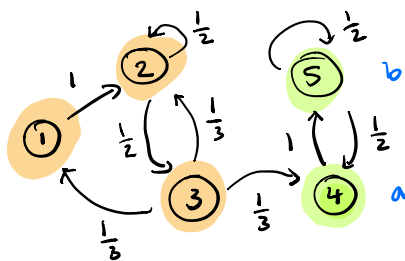
$$\alpha(i) = 0 \quad \forall i \in B$$

$$\alpha(i) = 1 \quad \forall i \in A$$

$$\alpha(i) = \sum_{k \in X} P(i, k) \alpha(k) \quad \text{o.w.}$$

- ① For each of the following chains, determine
- reducibility / irreducibility
 - periodicity / aperiodicity if it applies
 - stationary distribution if it is unique

Chain 1



reducible
periodicity DNA

$$\begin{bmatrix} 0 & 0 & 0 & a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

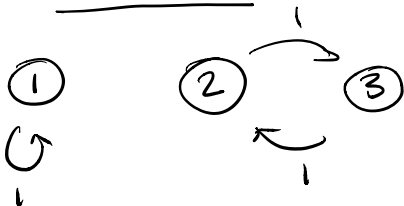
$$a + b = 1$$

$$a = \frac{b}{2} \Rightarrow a = \frac{1}{3}$$

$$a = a + \frac{1}{2}b - a \quad b = \frac{2}{3}$$

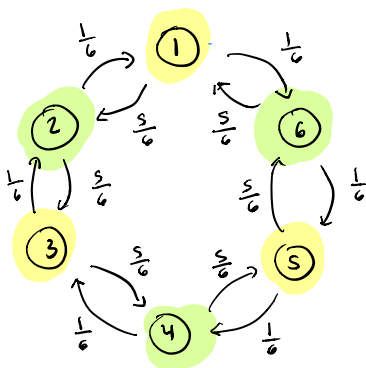
$$= \frac{1}{2}b$$

Chain 2



reducible
periodicity DNA
no unique s. d.

Chain 3

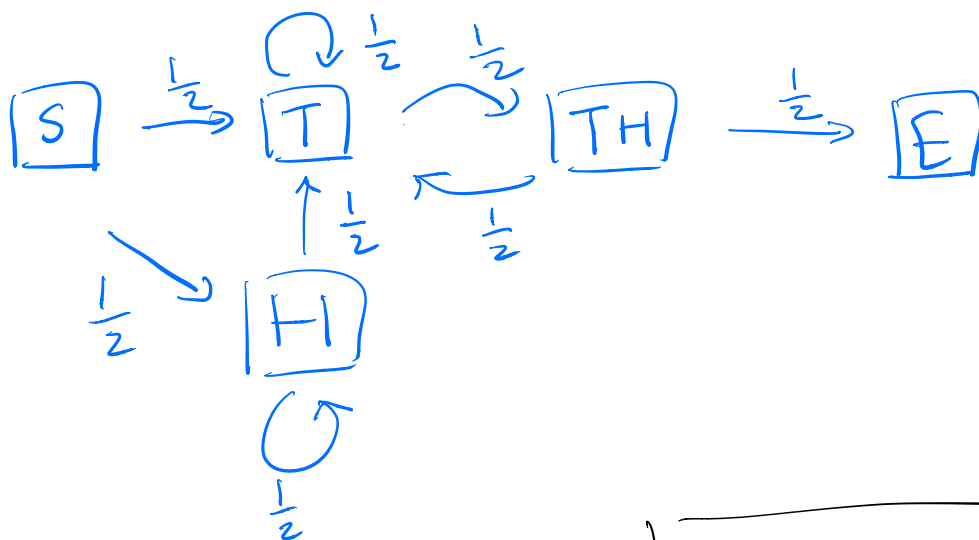


irreducible
period of 2

stationary dist:

$$\left[\frac{1}{6} \quad \frac{1}{6} \quad \dots \quad \frac{1}{6} \right]$$

- ② You flip a fair coin. What is the average number of **tails** until you get $\{ \frac{1}{2} \text{ tails, heads, heads} \}$? (Spring 2016 final 3.3)



$$g(E) = 0$$

$$g(S) = 4$$

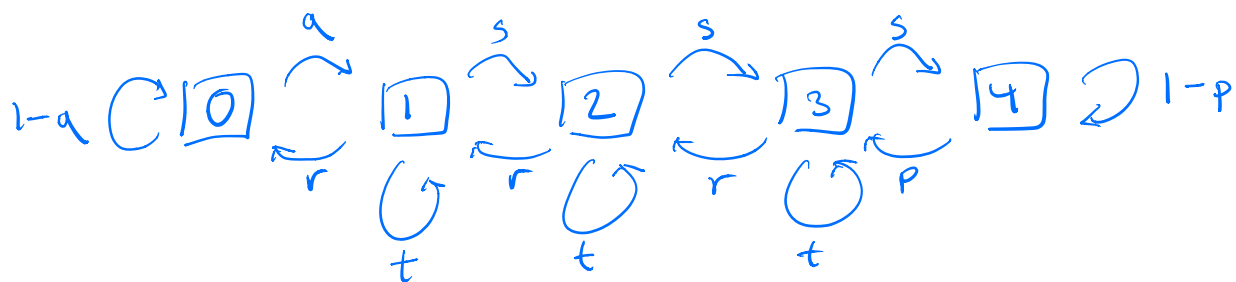
$$g(TH) = \frac{1}{2} (g(E) + 0) + \frac{1}{2} (g(T) + 1)$$

$$g(T) = \frac{1}{2} (g(TH) + 0) + \frac{1}{2} (g(T) + 1)$$

$$g(H) = \frac{1}{2} (g(H) + 0) + \frac{1}{2} (g(T) + 1)$$

$$g(S) = \frac{1}{2} (g(H) + 0) + \frac{1}{2} (g(T) + 1)$$

3. Customers are in line at a store. At any given time step, the customer at the front of the line leaves with probability p , and, independent of that, a new customer joins the line with probability q . The line can only handle 4 people at most, so no new customer will join if there are already 4 customers present in the line. If there are 2 customers currently in line, what is the probability the line is empty before the line contains 4 people?



Where $r = p(1-q)$, $s = (1-p)q$, $t = pq + (1-p)(1-q)$

Let $\alpha(i)$ be $P(\text{reach 0 before 4} \mid \text{starting @ } i)$

$$\alpha(0) = 1$$

$$\alpha(1) = r\alpha(0) + t\alpha(1) + s\alpha(2)$$

$$\alpha(2) = r\alpha(1) + t\alpha(2) + s\alpha(3)$$

$$\alpha(3) = r\alpha(2) + t\alpha(3) + s\alpha(4)$$

$$\alpha(4) = 0$$

Solve the system $\Rightarrow \alpha(z) = \frac{r^2}{(1-t)^2 - 2rs}$

$$= \frac{p^2(1-q)^2}{(p+q-2pq)^2 - 2p(1-p)q(1-q)}$$