Quick Review:

Cardinality: a generalization of "size".

- Allows us to reason about / compare infinite sets
- Use injections / surjections instead of # of elements.

Strategies to show $|A| = |B|$:

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S is <u>countable</u> if it is either finite or $|S| = |\mathbb{N}|$.

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S is <u>uncountable</u> if it is not countable.

can be shown using <u>diagonalization</u>:

- Prove it using contradiction

(1)

(2)

Some Sets Whose Cardinalities You Should Know:

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Computability - Can you write a program that executes any given fn?

- No! # of functions is uncountable, while # of programs is countable
- Ex: TestHalt(P, x) is uncomputable.
  ↳ Common strat to show that P is uncomputable is to use P to solve TestHalt.

Unions and Intersections

2A #1

Decide if the following expressions are either "Always Countable," "Sometimes countable," "Always Uncountable," or "Sometimes Uncountable." Provide proof / examples.

(a) $A \cap B$, where A is countable and B is uncountable.

(b) $A \cup B$, where A is countable and B is uncountable.

(c) $\bigcap_{i \in A} S_i$, where A is a countable set of indices and S_i is uncountable for all i.

Counting Cartesian Products

2A # 2

(a) The Cartesian product of two sets A, B is

$$A \times B := \{(a,b) \mid a \in A, b \in B\}.$$

Prove that if A and B are countable, then $A \times B$ is countable.

(b) For all positive integers $n \geq 2$, prove that the set

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable when A_i is countable for all $1 \leq i \leq n$.

(c) Consider a countable collection of countable sets $B_1, B_2, \ldots$. Under what conditions is $B_1 \times B_2 \times \cdots$ a countable set? Provide proof.

Hello World

2A # 3

Determine if the following tasks are computable. Provide either a program or a proof of uncomputability.

(a) A program that takes in a program P and an input x, and determines whether P(x) prints "Hello World".

(b) A program that takes in a program P and an integer K and determines whether P prints "Hello World" before the Kth line of P is executed.

(c) A program that takes in a program P and an integer K, and determines whether P prints "Hello World" when the first K lines are executed.