Important Facts

- If $G = (V, E)$ is a graph, the following statements are equivalent. (Let $n = |V|$)
  (1) G is a tree.
  (2) G is connected and acyclic.
  (3) G is connected and has $n-1$ edges.
- If $G = (V, E)$ is a graph, the following are also equivalent.
  (1) G is complete.
  (2) G has $n(n-1)/2$ edges.
  (3) Every vertex in G has degree $n-1$.
- Hypercubes in n dimensions . . .
  - Have 2^n vertices
  - Have $n \cdot 2^{n-1}$ edges
  - Are bipartite

True or False: | 2B # 1

(a) Any pair of vertices in a tree are connected by exactly one path.

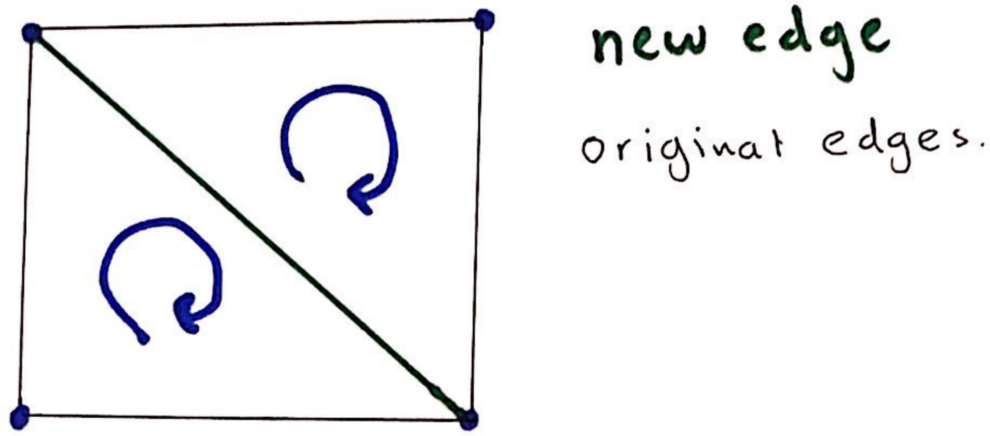
(b) Adding an edge between two vertices of a tree creates a cycle.

(c) Adding an edge in a connected graph creates exactly one new cycle.

(a) True. Let T be a tree and v_1, v_2 vertices. T is connected so there is a path between v_1 and v_2. If we have two paths, then we can generate a cycle, contradicted by the fact that T is a tree. See below

(b) True. Adding an edge introduces a new path, with again generates a cycle as above.

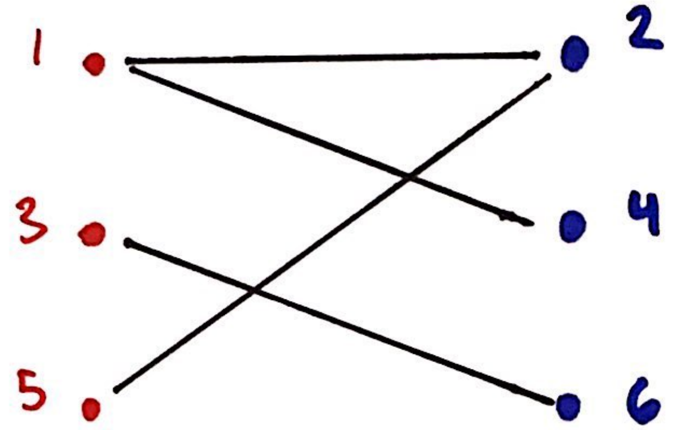
(c) False. See the counter example below:



The green edge generates 2 new cycles.

Bipartite Graph

A graph is bipartite if we can split its vertices into two nonempty sets such that every edge in the graph connects vertices in different sets.

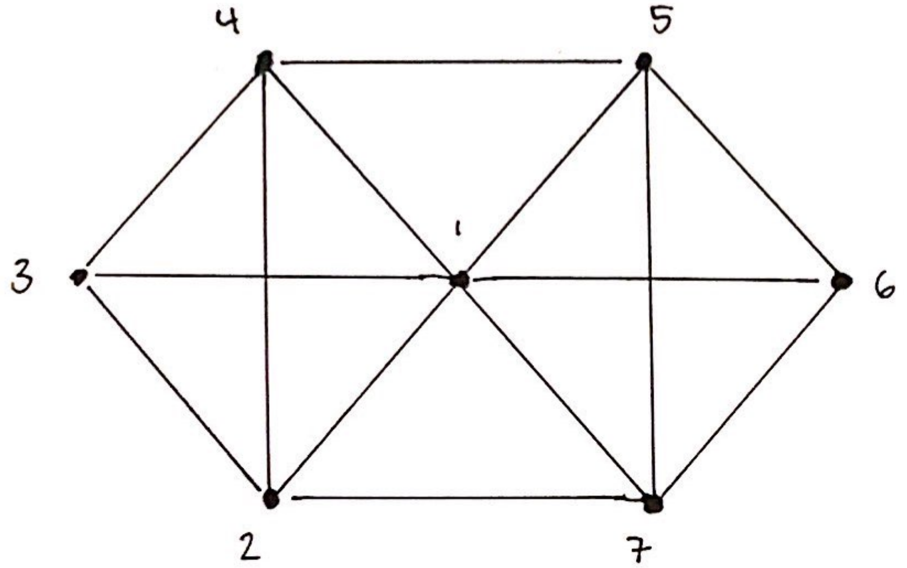


Prove that a bipartite graph contains no tours of odd length.

In order for a walk to be a tour, it needs to start and end at the same vertex. Observe that every walk in a bipartite graph must alternate between red and blue vertices. Thus in order to end up where we started, we need to take an even number of steps. Thus every tour must have an even length.

Eulerian Tour / Walk

2B # 3



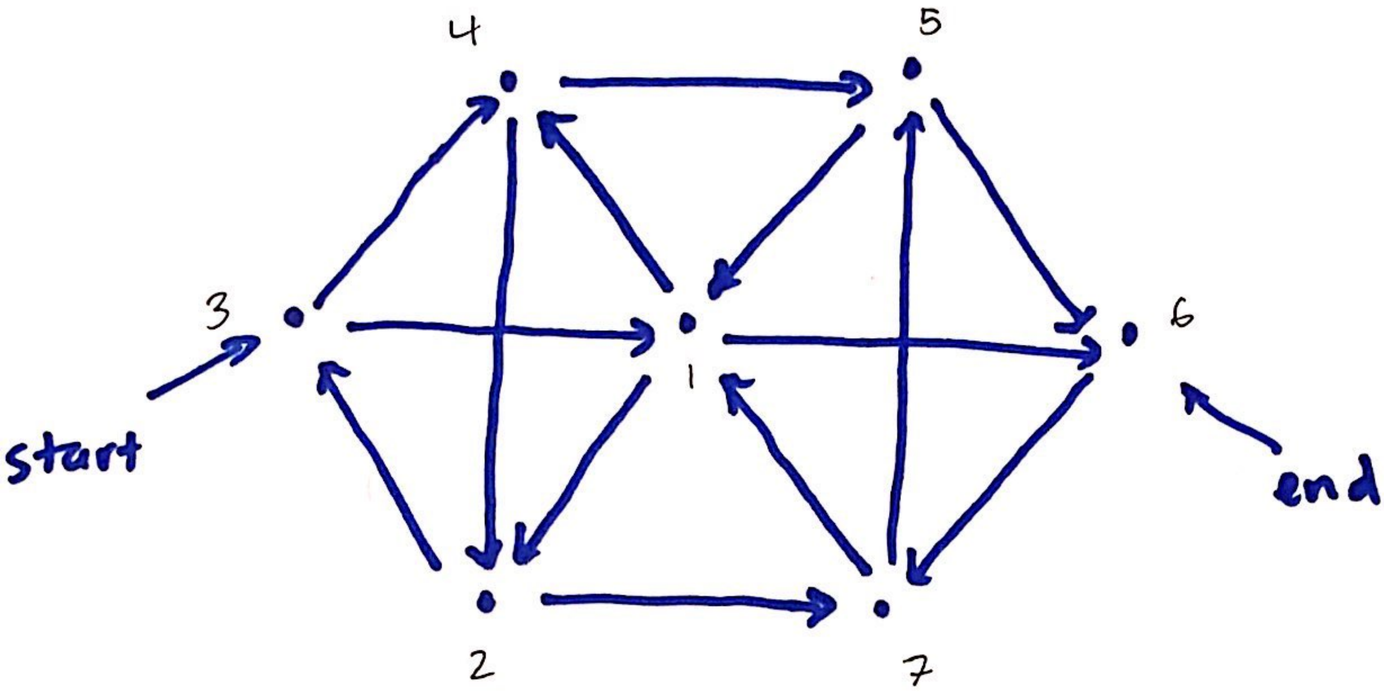
(a) Does the graph have an Euler Tour?

No, vertices 3 and 6 have odd degree.

(b) Does it have an Euler Walk?

Yes.

$3 \to 4 \to 2 \to 3 \to 1 \to 4 \to 5 \to 1 \to 2 \to 7 \to 1 \to 6 \to 7 \to 5 \to 6$


4
5
3
1
6
start
end
2
7

Odd Degree Vertices

2B # 4

Let $G = (V, E)$ be an undirected graph. Prove that G has an even number of vertices with odd degree.

Suppose G has an odd number of vertices w/ odd degree. We know that

$$\sum_{v \in V} \deg(v) = 2|E| \text{ is even;}$$

since there are an odd number of vertices with od d degree, this means that the sum of the degrees of all vertices must be odd, contradiction.