Quick Review

Let G be a graph:

- G is connected, planar $\Rightarrow v - e + f = 2$
  - If $v \geq 3$, then $e \leq 3v - 6$
  - If $v \geq 3$ and G is bipartite, then $e \leq 2v - 4$
- (Kuratowski) G is nonplanar if and only if it contains something equivalent to K_5 or $K_{3,3}$.
- The chromatic number $\chi(G)$ is # of colors needed to color the vertices of G so that no adjacent vertices have the same color.
  - If G is planar, $\chi(G) \leq 5$
  - If H is a subgraph of G, $\chi(G) \geq \chi(H)$

General Notes:

- If you show a subgraph of G is nonplanar, then G is nonplanar.
- When inducting on graphs, be wary of build-up error and removal error.
- Don't be afraid to work with connected components individually instead of the whole graph.

Short Answer

2C #1

(a) Bob removed a degree 3 node from an n vertex tree. How many connected components does the resulting graph have?

(b) Starting with an n vertex tree, Bob adds 10 edges. Then, Alice removes 5 edges (not necessarily the ones Bob added). If the resulting graph has 3 connected components, how many more edges does Alice have to remove to eliminate all cycles?

(a) 3. The graph is a tree, so if we remove this vertex there is no path from one of its neighbours to another.

(b) 7. Suppose that the components have a, b, c vertices respectively. Then $a+b+c=n$. Now, in order for the graph to be acyclic, each of these components must be a tree, thus we need a total of $(a-1)+(b-1)+(c-1)=n-3$ edges. We start with $n-1$, add 10, then remove 5, so we have $n+4$. Thus we need to remove 7 more.

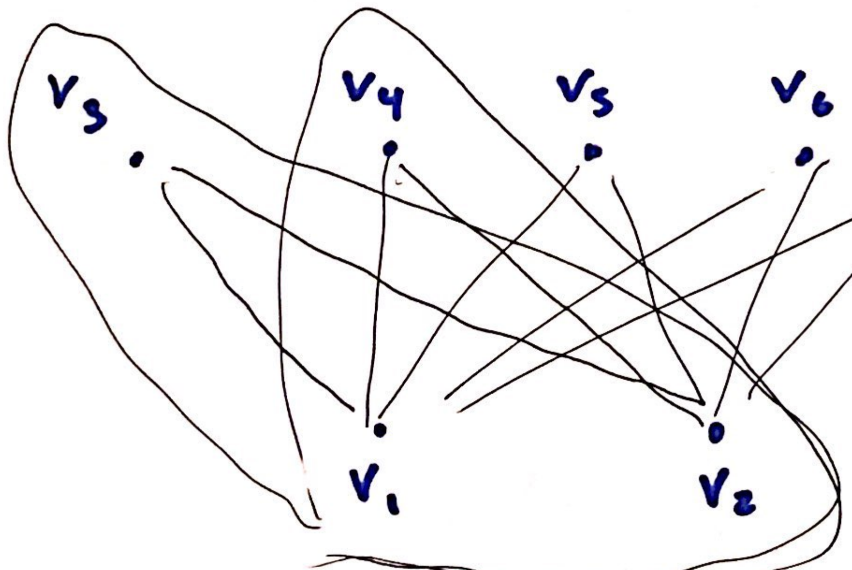
Planarity

2C # 2

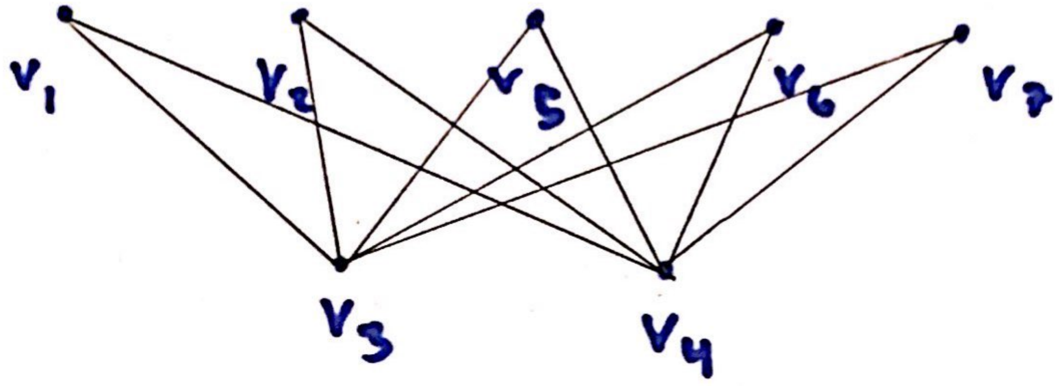
Let G be a graph with the property that for any 3 distinct vertices v_1, v_2, v_3, there are at least two edges between them. Prove that if G has ≥ 7 vertices, then G is not planar. (Hint: prove that if G doesn't contain K_5, then it contains $K_{3,3}$)

Idea: Assume G has 7 vertices and is planar. Then G doesn't contain K_5 — use this to prove G contains $K_{3,3}$.

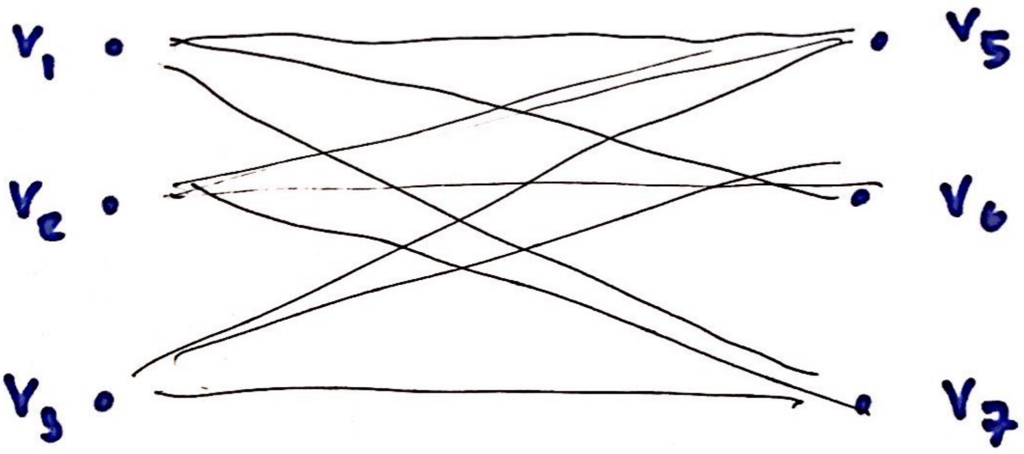
Proof: Let the vertices of G be $v_1, \ldots, v_7$. Since $K_5 \not\subseteq G$, there are vertices in $\{v_1, v_2, v_3, v_4, v_5\}$ that are not connected. WLOG let these be v_1, v_2.



Same argument applies to $\{v_3, v_4, v_5, v_6, v_7\}$
WLOG, let v_3, v_4 be unconnected.



Now, lets look at what edges we know are in G (note I didn't draw v_4, we don't need it)



This means that G contains $K_{3,3}$, so its not planar.

Scanned with CamScanner

Graph Coloring

2C #3

Prove that a graph with maximum degree K is (K+1) colorable.

Idea: induction on # of vertices.

Base Case: n=1 is clearly 0+1=1 colorable.

Assume all m-vertex graphs with maximum degree K are (K+1)-colorable, and consider an m+1 degree graph with max degree K.

Select an arbitrary vertex v. We know that $\deg(v) \leq K$ by the conditions we start with. Moreover, if we remove v, we always get a graph with m vertices and maximum degree $\leq K$, so we can color it with K+1 colors. Now, when we add back v, we can always find a color for it as we have K+1 to choose from and v is connected to at most K other vertices, thus the graph is (K+1) colorable.

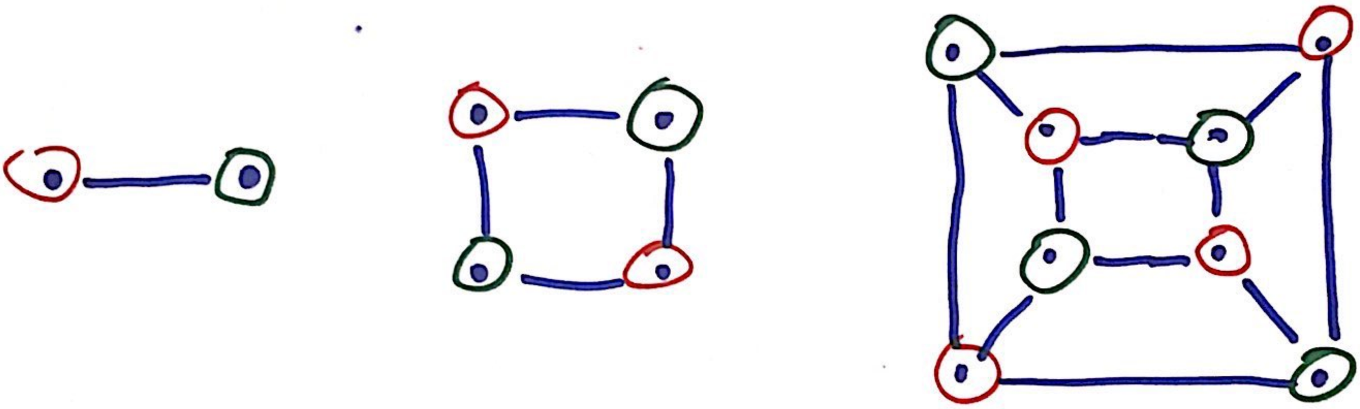
Hypercubes:

2C # 4

Recall that an n-dimensional hypercube is a graph generated by n-bit strings such that

- Each vertex corresponds to a unique string of n 0s and 1s,
- Two vertices are connected if their bit strings differ at exactly one place.

Prove that for all n, the n-dimensional hypercube is bipartite. (Hint: draw out small hypercubes and look for a pattern)



Idea: induct on n.

Assume k-d hypercubes are bipartite. we construct (k+1)-d hypercubes by taking two copies of a k-d hypercube and joining corresponding edges. Each of these is bipartite so partition them and flip the colors on one of them. This is a valid partition, so we're done.